Lecture 2

- Fubini's thm (cont'd)
- Properties of Integral
- Piecewise Continuous Functions

Last time we formulated the Fubini's thm.

<u>Fubini's Thm</u> Let f be continuous : $R = [a,b] \times [c,d]$. Then

$$\iint_R f = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

Sketchy Pf : When $\|P\|$ is v. small,

error

$$\iint_R f - \sum_{j=1}^{n} \sum_{k=1}^{m} f(P_{jk}) \Delta x_j \Delta y_k = E, \quad \begin{array}{c} E \to 0 \\ \text{as } \|P\| \to 0 \end{array}$$

Choose tag pts $P_{jk} = (x_j^*, y_k^*)$, $a = x_0 < x_1 < \cdots < x_n = b$, $x_j^* \in [x_{j-1}, x_j]$
$c = y_0 < y_1 < \cdots < y_m = d$ $y_k^* \in [y_{k-1}, y_k]$

$$\iint_R f = \sum_{j=1}^{n} \sum_{k=1}^{m} f(x_j^*, y_k^*) \Delta y_k \Delta x_j$$

$$= \sum_{j=1}^{n} \left[\sum_{k=1}^{m} f(x_j^*, y_k^*) - \int_c^d f(x_j^*, y)\,dy \right] \Delta x_j$$

$$+ \sum_{j=1}^{n} \left(\int_c^d f(x_j^*, y)\,dy \right) \Delta x_j + E$$

Since the first term $\to 0$ as $\|P\| \to 0$, it is an error term, absorb it to E.

$$\iint_R f = \sum_{j=1}^{n} \left(\int_c^d f(x_j^*, y)\, dy \right) \Delta x_j + E$$

Let $G(x) = \int_c^d f(x,y)\,dy$ which is conti on $[a,b]$.

$$\iint_R f = \sum_{j=1}^{n} G(x_j^*) \Delta x_j + E$$

$$= \left(\sum_{j=1}^{n} G(x_j^*) \Delta x_j - \int_a^b G(x)\,dx \right) + \int_a^b G(x)\,dx + E.$$

The first term $\to 0$ as $\|P\| \to 0$, so

$$\iint_R f = \int_a^b G(x)\,dx + E$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx + E$$

$$\to \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

after letting $\|P\| \to 0$.

Properties of integrals.

<u>theorem 4</u> (linearity) f, g integrable on R

$\Rightarrow \alpha f + \beta g$ integrable in R, and

$$\iint_R (\alpha f + \beta g) = \alpha \iint_R f + \beta \iint_R g .$$

Pf : Divide in two parts ① $\iint \alpha f = \alpha \iint f$

② $\iint (f+g) = \iint f + \iint g$.

Just show ②. We've, letting

$$I = \iint_R f + \iint_R g ,$$

$$\left| \sum_{j,k} (f+g)(P_{jk}) \Delta x_j \Delta y_k - I \right|$$

$$= \left| \sum_{j,k} (f+g)(P_{jk}) \Delta x_j \Delta y_k - \iint_R f - \iint_R g \right|$$

$$\leq \left| \sum_{j,k} f(P_{jk}) \Delta x_j \Delta y_k - \iint_R f \right| + \left| \sum_{j,k} g(P_{jk}) \Delta x_j \Delta y_k - \iint_R g \right|$$

As f, g integrable, $\forall \varepsilon > 0$, $\exists \delta$ s.t. $\forall P$, $\|P\| < \delta$,

$$\left| \sum f(P_{jk}) \Delta x_j \Delta y_k - \iint_R f \right| < \varepsilon$$

$$\left| \sum g(P_{jk}) \Delta x_j \Delta y_k - \iint_R g \right| < \varepsilon$$

So

$$\left| \sum (f+g)(P_{jk}) \Delta x_j \Delta y_k - I \right| < 2\varepsilon, \text{ done.} \quad \square$$

This theorem holds for integrals in all dimensions.

Let
$$V = \{\text{all integrable functions on } R\}$$

Theorem 4 tells us that it is a vector space. So,

$$T : V \longrightarrow \mathbb{R}$$
$$Tf \overset{\text{def}}{=} \iint_R f$$

becomes a linear map (linear transformation).

<u>Theorem 5</u> (positivity) f integrable, $f \geq 0$. then

$$\iint_R f \geq 0 .$$

Pf: Obvious (by looking at $S(f, \mathring{P})$ and $\|P\| \to 0$)

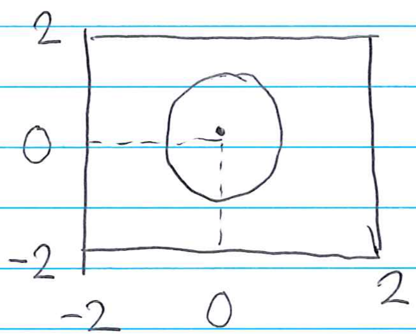
Equivalently, f, g integrable and $f \geq g$. Then

$$\iint_R f \geq \iint_R g .$$

Fubini's Thm not only holds for continuous functions. It also holds for "piecewise continuous functions". This is a generalization of "continuous fcn except at finitely many points" in the 1-dim case.

A function in R is a piecewise continuous fcn if it is continuous except at finitely many points or finitely many piecewise smooth curves.

e.g.

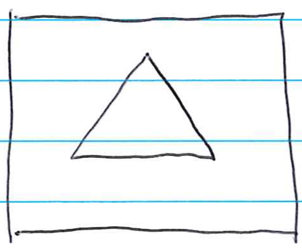


$$f(x,y) = \begin{cases} x^2 y, & (x,y),\ x^2+y^2 \leq 1 \\ 1, & (x,y),\ x^2+y^2 > 1 \end{cases}$$

$(x,y) \in R = [-2,2] \times [-2,2]$.

f conti in $x^2+y^2<1$, $x^2+y^2>1$, but not along $x^2+y^2=1$.

e.g.



$$g(x,y) = \begin{cases} x+y, & (x,y) \text{ inside } \Delta \\ 0, & (x,y) \text{ outside } \Delta \\ 1, & (x,y) \text{ on } \Delta \end{cases}$$

Theorem 3' Theorem 3 (Fubini thm) holds for piecewise continuous functions.